# Moment varieties for mixtures of products

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# The (nonparametric) set-up

Consider $n$ independent random variables $X_1, X_2, \ldots, X_n$ on the line $\mathbb{R}$.
*Assumptions:*

- ⋆ **No** assumptions about $X_k$, only that moments $\mu_{ki} = \mathbb{E}(X_k^i)$ exist.
- ⋆ The moments $\mu_{ki}$ are unknowns.
- ⋆ The only equations we require are $\mu_{k0} = 1$ for $k = 1, 2 \ldots, n$.

We consider a random variable $X$ on $\mathbb{R}^n$ that is the product of these $n$ arbitrary independent random variables on $\mathbb{R}$. By independence, we have

$$\mathbb{E}(X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}) \;=\; \mathbb{E}(X_1^{i_1}) \cdot \mathbb{E}(X_2^{i_2}) \cdots \mathbb{E}(X_n^{i_n}).$$

This leads us to the *moment variety* $\mathcal{M}_{n,d}$, which has parametrization

$$\boxed{m_{i_1 i_2 \cdots i_n} \;= \mu_{1i_1}\mu_{2i_2} \cdots \mu_{ni_n} \text{ where } i_1, i_2, \ldots, i_n \geq 0 \text{ and } i_1 + i_2 + \cdots + i_n = d.}$$

The image is a toric variety of dimension at most $nd - 1$ in $\mathbb{P}^{\binom{n+d-1}{d}-1}$.

# Example

Consider $\mathcal{M}_{5,3}$ in $\mathbb{P}^{34}$. The solutions to $i_1 + i_2 + i_3 + i_4 + i_5 = 3$ can be grouped into three partitions: $\lambda = (1\ 1\ 1)$, $\lambda = (2\ 1)$, $\lambda = (3)$. Consider the following three toric varieties of dimensions $4, 8, 4$ respectively:

$$\mathcal{M}_{5,(111)} \subset \mathbb{P}^9 : m_{11100} = \mu_{11}\mu_{21}\mu_{31}, \ldots, m_{00111} = \mu_{31}\mu_{41}\mu_{51}.$$
$$\mathcal{M}_{5,(21)} \subset \mathbb{P}^{19} : m_{21000} = \mu_{12}\mu_{21}, m_{12000} = \mu_{11}\mu_{22}, \ldots, m_{00012} = \mu_{41}\mu_{52}.$$
$$\mathcal{M}_{5,(3)} = \mathbb{P}^4 : m_{30000} = \mu_{13}, m_{03000} = \mu_{23}, \ldots, m_{00003} = \mu_{53}.$$

Combining these parametrizations yields the original variety.

We will also study $\mathcal{M}_{n,d}$ under projections $\mathbb{P}^{\binom{n+d-1}{d}-1} \dashrightarrow \mathbb{P}^{|N_\lambda|-1}$ for any partition $\lambda$ of $d$ with $\leq n$ parts. We denote these toric varieties by $\mathcal{M}_{n,\lambda}$.

# Toric combinatorics

First, we are interested in studying the toric varieties $\mathcal{M}_{n,d}$ and $\mathcal{M}_{n,\lambda}$.

*Familiar examples:*

- $\star$ For any $n$, consider the partition $\lambda = (1^d) = (1\,1\ldots 1)$ of $d < n$. Then $\mathcal{M}_{n,(1^d)}$ is the associated toric variety to the **hypersimplex**

  $$\Delta(n,d) = \operatorname{conv}\{e_{\ell_1} + e_{\ell_2} + \cdots + e_{\ell_d} \,:\, 1 \leq \ell_1 < \ell_2 < \cdots < \ell_d \leq n\}$$

  It has dimension $n-1$ in $\mathbb{P}^{\binom{n}{d}-1}$.
- $\star$ Consider the partition $\lambda = (n-1, n-2, \ldots, 2, 1)$. Then the moment variety $\mathcal{M}_{n,\lambda}$ is the toric variety of the **Birkhoff polytope**, which lives in $\mathbb{P}^{n!-1}$ and has dimension $(n-1)^2$.

# Toric results

**Theorem (A., Kileel, Sturmfels)**

*The dimension of the moment variety $\mathcal{M}_{n,d}$ is* $\min\left\{nd-1, \binom{n+d-1}{d}-1\right\}$.

Given a partition $\lambda$ of length $n$, let let $k_0 \geq k_1 \geq \ldots \geq k_s$ be multiplicities of the distinct parts in $\lambda$. We define

$$\nu = (\underbrace{s,\ldots,s}_{k_s}, \underbrace{s-1\ldots,s-1}_{k_{s-1}},\ldots,\underbrace{1,\ldots,1}_{k_1},\underbrace{0\ldots,0}_{k_0}).$$

to be the *reduction* of $\lambda$. Here $s+1$ is the number of distinct parts of $\lambda$.

**Ex**: both $(8,5,5,4)$ and $(7,7,3,0)$ reduce to $\nu = (2,1,0,0)$, with $s=2$.
**Ex**: if $\lambda = (1^d)$, we recover the identification $\Delta(n,d)$ with $\Delta(n,n-d)$.

**Theorem (A., Kileel, Sturmfels)**

*The moment variety $\mathcal{M}_{n,\lambda} = \mathcal{M}_{n,\nu}$ has dimension $(n-1)s$.*

# What about generators?

**Example**

Consider the variety $\mathcal{M}_{4,4}$ in $\mathbb{P}^{34}$. Its ideal is generated by 52 quadrics and 28 cubics. The subset of the generators which involves the twelve unknowns $m_{2110}, \ldots, m_{0112}$ does not suffice to cut out $\mathcal{M}_{4,(211)}$ in $\mathbb{P}^{11}$.

Building upon work of Yamaguchi, Ogawa, and Takemura *"Markov degree of the Birkhoff model"*:

**Theorem (A., Kileel, Sturmfels)**

*For any partition $\lambda$, the ideal of $\mathcal{M}_{n,\lambda}$ is generated by quadrics and cubics.*

The ideals for $\mathcal{M}_{n,d}$ are more complicated. We conjecture that there does not exist a uniform degree bound for their generators.

# Mixtures

Now we consider the mixtures of $r$ copies of our toric models. Algebraically, these are the *secant varieties* $\sigma_r(\mathcal{M}_{n,d})$ and $\sigma_r(\mathcal{M}_{n,\lambda})$. The first is parametrized by

$$\boxed{m_{i_1 i_2 \cdots i_n} = \sum_{j=1}^{r} \mu_{1i_1}^{(j)} \mu_{2i_2}^{(j)} \cdots \mu_{ni_n}^{(j)} \text{ with } i_1, i_2, \ldots, i_n \geq 0 \text{ and } i_1 + i_2 + \cdots + i_n = d.}$$

These varieties are no longer toric! What can we say about their dimensions, degrees, generators?

◇ Consider the secant variety $\sigma_2(\mathcal{M}_{5,2})$. The parametrization is given as

$$m_{20000} = \mu_{12}^{(1)} + \mu_{12}^{(2)} \ , \ \ldots \ , \ m_{11000} = \mu_{11}^{(1)} \mu_{21}^{(1)} + \mu_{11}^{(2)} \mu_{21}^{(2)}, \ \ldots .$$

# Example (continued)

Note $\mathcal{M}_{5,2} = \mathcal{M}_{5,(2)} \star \mathcal{M}_{5,(11)} = \mathbb{P}^4 \star \mathcal{M}_{5,(11)}$, since

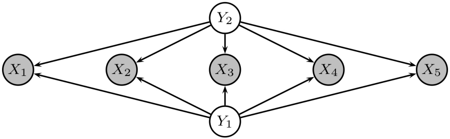
$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{12} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{13} & m_{23} & m_{33} & m_{34} & m_{35} \\ m_{14} & m_{24} & m_{34} & m_{44} & m_{45} \\ m_{15} & m_{25} & m_{35} & m_{45} & m_{55} \end{bmatrix} = \begin{bmatrix} \mu_{12} & \mu_{11}\mu_{21} & \mu_{11}\mu_{31} & \mu_{11}\mu_{41} & \mu_{11}\mu_{51} \\ \mu_{11}\mu_{21} & \mu_{22} & \mu_{21}\mu_{31} & \mu_{21}\mu_{41} & \mu_{21}\mu_{51} \\ \mu_{11}\mu_{31} & \mu_{21}\mu_{31} & \mu_{32} & \mu_{31}\mu_{41} & \mu_{31}\mu_{51} \\ \mu_{11}\mu_{41} & \mu_{21}\mu_{41} & \mu_{31}\mu_{41} & \mu_{42} & \mu_{41}\mu_{51} \\ \mu_{11}\mu_{51} & \mu_{21}\mu_{51} & \mu_{31}\mu_{51} & \mu_{41}\mu_{51} & \mu_{52} \end{bmatrix}.$$

$$\sigma_2(\mathcal{M}_{5,2}) = \sigma_2\left(\mathbb{P}^4 \star \mathcal{M}_{5,(11)}\right) = \mathbb{P}^4 \star \boxed{\sigma_2(\mathcal{M}_{5,(11)})} \subset \mathbb{P}^4 \star \mathbb{P}^9 = \mathbb{P}^{14}.$$

The ideal of $\sigma_2(\mathcal{M}_{5,(11)})$ is principal, generated by the pentad

$$\begin{aligned} & m_{12}m_{13}m_{24}m_{35}m_{45} - m_{12}m_{13}m_{25}m_{34}m_{45} - m_{12}m_{14}m_{23}m_{35}m_{45} + m_{12}m_{14}m_{25}m_{34}m_{35} \\ &+ m_{12}m_{15}m_{23}m_{34}m_{45} - m_{12}m_{15}m_{24}m_{34}m_{35} + m_{13}m_{14}m_{23}m_{25}m_{45} - m_{13}m_{14}m_{24}m_{25}m_{35} \\ &- m_{13}m_{15}m_{23}m_{24}m_{45} + m_{13}m_{15}m_{24}m_{25}m_{34} + m_{14}m_{15}m_{23}m_{24}m_{35} - m_{14}m_{15}m_{23}m_{25}m_{34}. \end{aligned}$$

This is the factor analysis model $\mathbf{F}_{5,2}$.




[*] Drton, Sturmfels, and Sullivant

# Dimensions of mixtures

## Proposition (A., Kileel, Sturmfels)

*The dimension of the moment variety satisfies the upper bound*

$$\dim\big(\sigma_r(\mathcal{M}_{n,d})\big) \;\leq\; \min\Big\{\, rnd - rn + n - 1,\ \tbinom{n+d-1}{d} - 1 \,\Big\}.$$

Because $\sigma_r(\mathcal{M}_{n,d}) = \sigma_r(\mathcal{M}_{n,(d)} \star \widetilde{\mathcal{M}}_{n,d}) = \mathbb{P}^{n-1} \star \sigma_r(\widetilde{\mathcal{M}}_{n,d})$.

## Theorem (A., Kileel, Sturmfels)

*The dimension $\sigma_r(\mathcal{M}_{n,d})$ is bounded above by the optimal value of*

$$\boxed{\begin{array}{ll} \text{maximize } c_1 + c_2 + \cdots + c_d - 1 \qquad & \text{subject to} \quad 0 \,\leq\, c_i \,\leq\, nr \quad \text{for} \;\; i \in [d] \\ & \text{and } \sum_{i \in S} c_i \,\leq\, \sum_{\lambda \cap S \neq \emptyset} |N_\lambda| \quad \text{for} \;\; S \subseteq [d]. \end{array}}$$

*The last sum ranges over partitions $\lambda \vdash d$ of length $\leq n$ having nonempty intersection with $S$.*

Conjecture: this bound is tight for $d \geq 3$!

# Implicitization

Solving the implicitization problem is difficult!

Consider the variety $\mathcal{M}_{6,(111)}$. Its ideal is given by the $2 \times 2$ minors of

$$\begin{bmatrix}
\star & \star & \star & \star & \star & m_{123} & m_{124} & m_{125} & m_{126} & m_{134} & m_{135} & m_{136} & m_{145} & m_{146} & m_{156} \\
\star & m_{123} & m_{124} & m_{125} & m_{126} & \star & \star & \star & \star & m_{234} & m_{235} & m_{236} & m_{245} & m_{246} & m_{256} \\
m_{123} & \star & m_{134} & m_{135} & m_{136} & \star & m_{234} & m_{235} & m_{236} & \star & \star & \star & m_{345} & m_{346} & m_{356} \\
m_{124} & m_{134} & \star & m_{145} & m_{146} & m_{234} & \star & m_{245} & m_{246} & \star & m_{345} & m_{346} & \star & \star & m_{456} \\
m_{125} & m_{135} & m_{145} & \star & m_{156} & m_{235} & m_{245} & \star & m_{256} & m_{345} & \star & m_{356} & \star & m_{456} & \star \\
m_{126} & m_{136} & m_{146} & m_{156} & \star & m_{236} & m_{246} & m_{256} & \star & m_{346} & m_{356} & \star & m_{456} & \star & \star
\end{bmatrix}$$

The ideal of $\sigma_2(\mathcal{M}_{6,(111)})$ is generated by 20 cubics and 12 quintics. The ideal of $\sigma_3(\mathcal{M}_{6,(111)})$ has no quadrics or cubics, but contains a unique quartic. Computations in **Julia** reveal:

$$\deg(\sigma_2(\mathcal{M}_{6,(111)})) = 465 \quad \text{and} \quad \deg(\sigma_3(\mathcal{M}_{6,(111)})) = 80.$$

# More implicitization

## Proposition (A., Kileel, Sturmfels)

*The variety* $\sigma_2(\mathcal{M}_{5,3})$ *has dimension* 24 *and degree* 3225 *in* $\mathbb{P}^{34}$*. Its prime ideal is generated by* 313 *polynomials, namely* 10 *cubics,* 283 *quintics,* 10 *sextics and* 10 *septics. These are obtained by elimination from the ideal of* $3 \times 3$ *minors of the* $5 \times 15$ *matrix*

$$\begin{bmatrix} a_{23} & a_{24} & a_{25} & a_{34} & a_{35} & a_{45} & \star & \star & \star & \star & \star & b_{21} & b_{31} & b_{41} & b_{51} \\ a_{13} & a_{14} & a_{15} & \star & \star & \star & a_{34} & a_{35} & a_{45} & \star & b_{12} & \star & b_{32} & b_{42} & b_{52} \\ a_{12} & \star & \star & a_{14} & a_{15} & \star & a_{24} & a_{25} & \star & a_{45} & b_{13} & b_{23} & \star & b_{43} & b_{53} \\ \star & a_{12} & \star & a_{13} & \star & a_{15} & a_{23} & \star & a_{25} & a_{35} & b_{14} & b_{24} & b_{34} & \star & b_{54} \\ \star & \star & a_{12} & \star & a_{13} & a_{14} & \star & a_{23} & a_{24} & a_{34} & b_{15} & b_{25} & b_{35} & b_{45} & \star \end{bmatrix}.$$

## Proposition (A., Kileel, Sturmfels)

*The variety* $\sigma_2(\mathcal{M}_{4,4})$ *has dimension* 27 *and degree* 8650 *in* $\mathbb{P}^{34}$*. Its prime ideal has only three minimal generators in degrees at most six.*

# Finiteness up to symmetry

Our ideals satisfy natural inclusions

$$I(\sigma_r(\mathcal{M}_{n,\bullet})) \subset I(\sigma_r(\mathcal{M}_{n+1,\bullet})), \qquad \text{where} \quad \bullet \in \{d, \lambda\},$$

by appending a zero to the indices of every coordinate: $m_{i_1 i_2 \cdots i_n} \mapsto m_{i_1 i_2 \cdots i_n 0}$.
Iterate these inclusions and let the big symmetric group act:

$$\langle S_n I(\sigma_r(\mathcal{M}_{n_0,\bullet}))\rangle \subseteq I(\sigma_r(\mathcal{M}_{n,\bullet})) \quad \text{for } n > n_0.$$

*Ideal-theoretic finiteness* means $\exists n_0$ such that equality holds for $n > n_0$.

**Theorem (A., Kileel, Sturmfels)**

*Given any partition $\lambda \vdash d$ and integer $r \geq 1$, set-theoretic finiteness holds for the varieties $\sigma_r(\mathcal{M}_{n,d})$ and $\sigma_r(\mathcal{M}_{n,\lambda})$. Ideal-theoretic finiteness holds in the toric case $r = 1$.*

Builds upon the results of Draisma and others.

# Example

The ideal of the variety $\mathcal{M}_{n,(1^d)}$ is generated by quadrics. The indices occurring in each quadratic binomial are 1 in at most $2d$ of the $n$ coordinates. Therefore, ideal-theoretic finiteness holds with $n_0 = 2d$.

If $\lambda = (1\,1)$, then $n_0 = 4$. Indeed:

$$
\begin{aligned}
I(\mathcal{M}_{4,\lambda}) = \langle & m_{0101}m_{1010} - m_{0110}m_{1001}, m_{0011}m_{1100} - m_{0110}m_{1001}\rangle \\
I(\mathcal{M}_{5,\lambda}) = \langle & m_{01001}m_{10100} - m_{01100}m_{10001}, m_{00011}m_{10100} - m_{00110}m_{10001}, \\
& m_{11000}m_{00101} - m_{01100}m_{10001}, m_{10010}m_{00101} - m_{00110}m_{10001}, \\
& m_{10010}m_{01100} - m_{10100}m_{01010}, m_{00011}m_{01100} - m_{00101}m_{01010}, \\
& m_{00110}m_{11000} - m_{10100}m_{01010}, m_{00011}m_{11000} - m_{10001}m_{01010}, \\
& m_{01001}m_{10010} - m_{10001}m_{01010}, m_{00110}m_{01001} - m_{00101}m_{01010}\rangle
\end{aligned}
$$

## Corollary

*Fix a partition $\lambda$ with $e$ nonzero parts, and suppose that $n$ increases. The toric varieties $\mathcal{M}_{n,\lambda}$ satisfy ideal-theoretic finiteness for some $n_0 \leq 3e$.*

*Thanks!*