Combinatorial Nullstellensatz: Various Proofs, Extensions and Applications

Yulia Alexandr
Advised by: Professor Karen Collins

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Hilbert's Nullstellensatz

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Definition

If I is an ideal, define:

$$\sqrt{I} = \{a : a^k \in I,\ k \in \mathbb{N}^+\}.$$

An ideal I is a *radical ideal* if $\sqrt{I} = I$

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- Take the ideal I to be the ideal generated by the polynomials $g_1(x_1), \cdots, g_n(x_n)$.

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- For all $i \in [n]$, define a special univariate polynomial $g_i(x_i)$.
- Take the ideal I to be the ideal generated by the polynomials $g_1(x_1), \cdots, g_n(x_n)$.
- The new ideal I is radical and so $f \in I$.
- Moreover, $\mathbb{F}$ may not be algebraically closed.

Combinatorial Nullstellensatz I

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Theorem (Combinatorial Nullstellensatz I)

Let $\mathbb{F}$ be a field and let $f = f(x_1, \cdots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \cdots, x_n]$. Let $A_1, \cdots, A_n$ be finite non-empty subsets of $\mathbb{F}$ and define $g_i(x_i) = \prod_{a \in A_i}(x_i - a)$. If $f(a_1, \cdots, a_n) = 0$ for all $a_i \in A_i$, then there are polynomials $h_1, \cdots, h_n \in \mathbb{F}[x_1, \cdots, x_n]$ such that:

$$f = \sum_{i=1}^{n} h_i g_i.$$

Combinatorial Nullstellensatz I Proofs

- Alon's original proof is constructive and involves analysis of polynomial roots.

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- Vishnoi's proof is purely algebraic and uses basic concepts in commutative algebra.

Combinatorial Nullstellensatz II

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Theorem (Combinatorial Nullstellensatz II)

Let $\mathbb{F}$ be a field and $f = f(x_1, \cdots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \cdots, x_n]$. For each $i \in [n]$, let t_i be a non-negative integer, and suppose $deg(f) = \sum_{i=1}^{n} t_i$. Also, suppose that the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is non-zero. Then, for all subsets $A_i \subseteq \mathbb{F}$ such that $|A_i| > t_i$, $i \in [n]$, there are $a_1 \in A_1, \cdots, a_n \in A_n$ such that $f(a_1, \cdots, a_n) \neq 0$.

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- Alon's proof uses the Combinatorial Nullstellensatz I to analyze polynomial degrees.
- Michałek's proof is independent of the Combinatorial Nullstellensatz I and uses induction on $\deg(f)$.

Generalized Combinatorial Nullstellensatz II

- Let $\mathbb{F}$ be an arbitrary field and let $f \in \mathbb{F}[x_1, \cdots, x_n]$ where $n \in \mathbb{N}^+$.

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- Define the **support** of f, denoted by $S(f)$, to be the set of all $(t_1, \cdots, t_n) \in \mathbb{N}^n$ such that the coefficient of $x_1^{t_1} \cdots x_n^{t_n}$ is non-zero in f.

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- Define a natural partial order on the set $S(f)$ by letting $(t_1, \cdots, t_n) \leq (s_1, \cdots, s_n)$ if and only if $t_i \leq s_i$ for all $i \in [n]$.

Generalized Combinatorial Nullstellensatz II

Theorem (Łason)

Let $\mathbb{F}$ be a field and $f \in \mathbb{F}[x_1, \cdots, x_n]$. Let $(t_1, \cdots, t_n) \in S(f)$ be a maximal element in $S(f)$. Then, for any subsets $A_i \subseteq \mathbb{F}$ such that $|A_i| \geq t_i + 1$ for all $i \in [n]$, there are $a_1 \in A_1, \cdots, a_n \in A_n$ such that $f(a_1, \cdots, a_n) \neq 0$.

Existing Applications

Cauchy-Davenport Theorem: the "Classical" Application

Definition

For any two subsets A and B of a field $\mathbb{F}$, we define their sum as follows:

$$A + B = \{a + b : a \in A, b \in B\}.$$

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Theorem (Cauchy-Davenport)

Let p be a prime and let A and B be two non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$. Then:

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

Proof 1 (Idea)

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Proof 1 (Idea)

- By induction on $|A|$.
- Uses counting arguments and basic group theory facts.

Proof 2

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Proof 2 via Combinatorial Nullstellensatz II

- We first claim the theorem holds when $|A|+|B| > p$.

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- Then there are some $b \in B$ and $a \in A$ such that $q-b=a$.

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- Hence, $q = a+b \in A+B$.
- Since q was arbitrary, $A+B = \mathbb{Z}/p\mathbb{Z}$.
- So, $|A+B| = p$ and the theorem holds.

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- Then, there exists some $C \subseteq \mathbb{Z}/p\mathbb{Z}$ such that $A+B \subseteq C$ and $|C| = |A|+|B|-2$.

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  - The coefficient of $x^{t_1}y^{t_2}$ in f is $\binom{|A|+|B|-2}{|A|-1}$, which is non-zero in $\mathbb{Z}/p\mathbb{Z}$.
- By the Combinatorial Nullstellensatz II, we get a contradiction. $\square$

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- To each vertex of $v \in V(G)$, associate a variable x_v.
- Define the **graph polynomial** f_G of G as follows:

$$f_G(x_1, x_2, \cdots, x_n) = \prod_{\substack{i<j \\ \{v_i, v_j\} \in E(G)}} (x_i - x_j).$$

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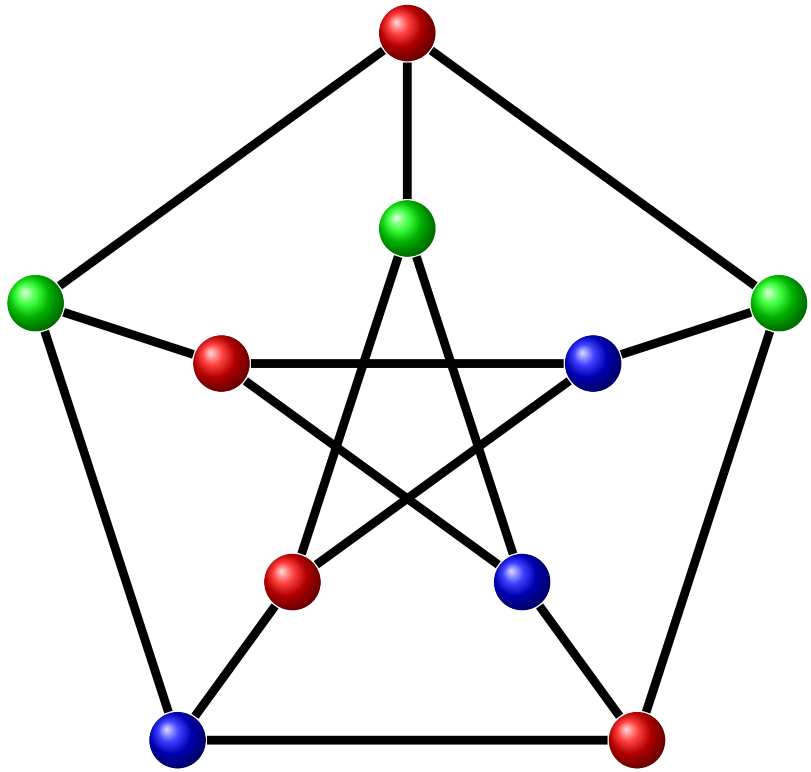
- A **vertex coloring** of a graph $G = (V, E)$ is a map $c : V \to C$ where C is a set of colors.
- A **proper** vertex coloring is a vertex coloring such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E(G)$.
- A graph G is k-**colorable** if there exists a proper coloring of G that uses k colors or less.

Example: the Petersen Graph

- A proper coloring of the Petersen Graph:
- The Petersen Graph is 3-colorable.
- It can be proven that the Petersen graph is not 2-colorable.



Graph Coloring

Theorem (Alon)

A graph $G = (V, E)$ is not k-colorable if and only if the graph polynomial f_G lies in the ideal generated by $\{x_v^k - 1 : v \in V\}$.

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Proof

Recall the graph polynomial is defined as:

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- Note that any coloring c of G gives an evaluation of the polynomial f_G, namely $f_G(c(x_1), \cdots, c(x_n))$.
- G is not k-colorable, so any coloring of its vertices with the kth roots of unity has two adjacent vertices sharing the same color.

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- Note that any coloring c of G gives an evaluation of the polynomial f_G, namely $f_G(c(x_1), \cdots, c(x_n))$.
- G is not k-colorable, so any coloring of its vertices with the kth roots of unity has two adjacent vertices sharing the same color.
- Then the graph polynomial f_G vanishes for any assignment of elements in A^n to $(x_1, \cdots, x_n)$.

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- Hence, it vanishes at all common zeros of g_v for all $v \in$ V.

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A graph $G = (V, E)$ is not k-colorable if and only if the graph polynomial f_G lies in the ideal generated by $\{x_v^k - 1 : v \in V\}$.

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- Thus, every coloring of f_G with the kth roots of unity makes f_G vanish.
- Hence, for any such coloring, there is an edge whose adjacent vertices are colored the same.
- So G is not k-colorable. $\square$

New Applications

Hypergraph-related Definitions

Definitions

1. A **hypergraph** $H = (V, E)$ is the finite set V (vertices) and a collection E of non-empty subsets of vertices (hyperedges).

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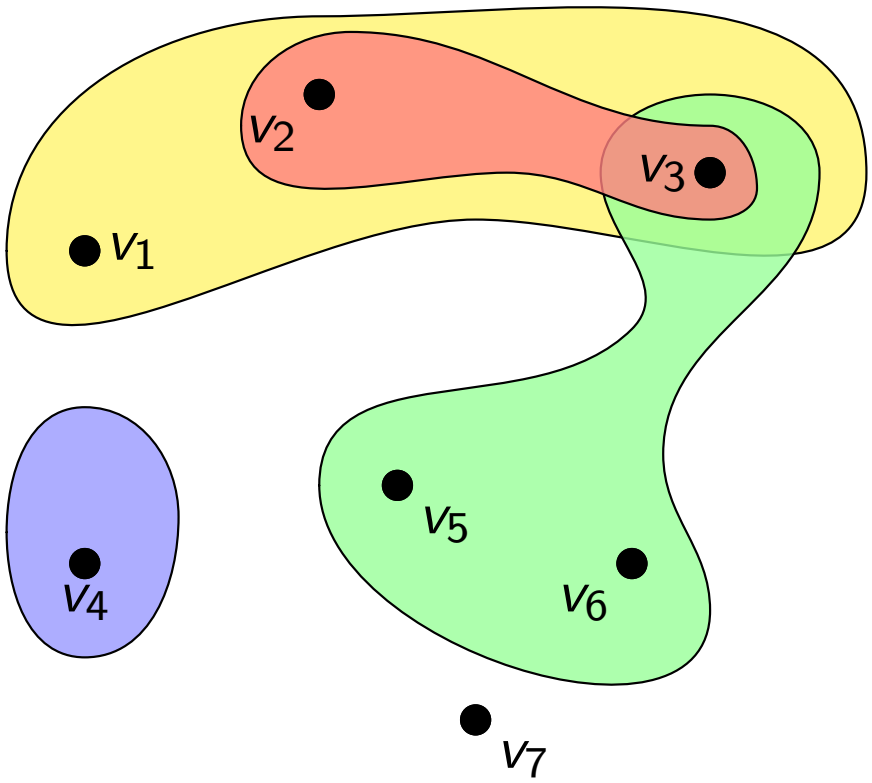
1. A **hypergraph** $H = (V, E)$ is the finite set V (vertices) and a collection E of non-empty subsets of vertices (hyperedges).
2. A hypergraph is m-**uniform** for some positive integer m if each hyperedge has cardinality m.
3. We say that a hypergraph H is k-**colorable** if there exists a coloring of its vertices with k or less colors such that no hyperedge is monochromatic.

Example of a Hypergraph



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Theorem (Alon)

A 3-uniform hypergraph $H = (V, E)$ is not 2-colorable if and only if the polynomial

$$g_H = \prod_{e \in E} \left(\left(\sum_{v \in e} x_v \right)^2 - 9 \right)$$

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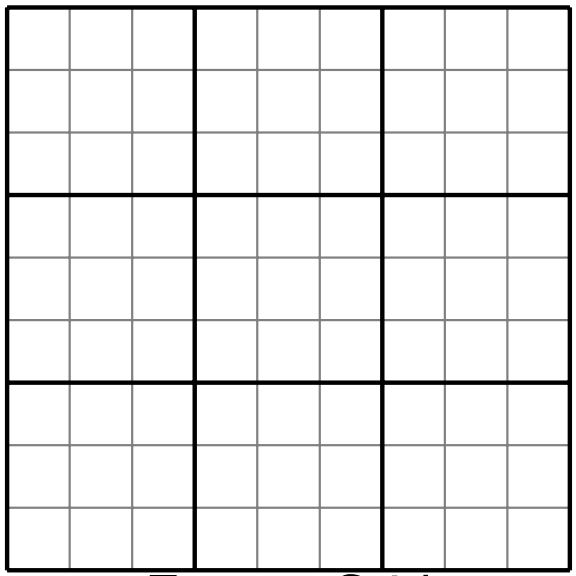
- Therefore, H is not 2-colorable. $\square$

Sudoku



Empty Grid

Sudoku

			4					
	5		7					9
			9			6		1
9	6	4					8	
		7				9		
3	1				2			
			8					7
2						1		3

Grid with Restrictions

Sudoku

1	8	9	4	2	6	7	3	5
6	5	2	7	1	3	8	4	9
7	4	3	9	8	5	6	2	1
9	6	4	1	5	7	3	8	2
5	2	7	3	4	8	9	1	6
8	3	1	2	6	9	5	7	4
3	1	5	6	7	2	4	9	8
4	9	6	8	3	1	2	5	7
2	7	8	5	9	4	1	6	3

Completed Puzzle

Sudoku Graph

- We define the **Sudoku graph** S by associating a vertex to each cell in the grid and placing an edge between two vertices if and only if they are in the same *row* or the same *column* or the same *block*.

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- S has 81 vertices and each cell/row/block is associated to a complete subgraph on 9 vertices.
- There are 27 such subgraphs in total; denote each of them by H_i, where $i \in [27]$.

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- Let S_R denote the graph S with the partial vertex coloring that corresponds to the restrictions, denoted by R, of a given puzzle.
- A puzzle with restrictions R has solutions if and only if S_R is 9-colorable with the colors in $[9]$.

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- We modify the polynomials repeatedly for all restrictions, keeping all the changes in previous steps.
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- After doing so for all vertices, let f_i be the new modified polynomials for all $i \in [27]$.

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Proof

Exercise.

Other Applications

- Minimum Bandwidth of a Graph
- f-choosability of Graphs
- Lucky Labeling

Questions?